Proof Complexity of Propositional Model Counting

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Credits

The credit for my work and understanding on propositional model counting –

- **Sravanthi Chede** IIT Ropar, Rupnagar, India.
- **Leroy Chew** TU, Wien, Austria.

Talk Contents

Proof Systems

- A proof system $f : \Delta^* \to \Sigma^*$ for a language $L \in \Sigma^*$ is a polynomial-time computable function such that the range of f is L.
- For $x \in L$, if $f(w) = x$, then w is an f-proof of the fact that $x \in L$. $|w|$ is the length of the proof.
- **Soundness**: For any $x \in \Sigma^*$ and $w \in \Delta^*$, if $f(w) = x$, then $x \in L$.
- **Completeness**: For every $x \in L$, there must exists $w \in \Delta^*$ such that $f(w) = x$.

Proof Systems

- Let f and g are two proof systems for a language L. we say that f **simulates** g if there is a computable function A that transforms the proofs in g to proofs in f with at most a polynomial blow in size.
- If A is polynomial time computable, then we say that f **p-simulates** g.
- f and g are p-equivalent if both can p-simulates each other.
- If f p-simulates g but g does not p-simulate f then we say that f is strictly stronger than g.
- We say that f and g are incomparable if both cannot p-simulates each other.
- Proof systems for the language UNSAT are called propositional poof systems. For example, the Resolution proof system.

Resolution (Res) Proof System [Blake 1937, Davis and Putnam 1960, Robinson 1965]

- Resolution rule: $\frac{(C \vee x) \quad (D \vee \neg x)}{(C \vee D)}$, here C and D are any clauses.
- Let F be an unsatisfiable CNF formula. A Resolution proof π of F is a sequence of clauses

$$D_1, D_2, \ldots, D_k$$

such that the last clause D_k is the empty clause and each D_i obeys one of the following

  - $D_i \in F$
  - D_i is derived from some clauses D_k, D_j, with $j, k < i$ via the resolution rule.

Reverse Unit Propogation [Goldberg and Novikov 2003]

- **Unit propagation (UP)**: Unit propagation satisfies the unit clauses of the CNF formula F by assigning their literal to true. Until you get a fix point or a conflict (x and $\neg x$ both become true for some variable x).
- Given an assignment α, $F|_\alpha$ denotes the CNF formula F' without the clauses of F satisfied by α and without the literals in the clauses of F falsified by α.
- Let F be a CNF formula and C a clause. Let α be the smallest assignment that falsifies C. We say that C is implied by F through UP (denoted $F \vdash_1 C$) if UP on $F|_\alpha$ results in a conflict.
- $F \vdash_1 C$ is known as Reverse Unit Propagation.

Propositional Model Counting

- For a given CNF formula F, the propositional model counting #SAT problem asks to compute the number of satisfying assignments.
- #SAT is one of the hardest known problems in the field of computational complexity.
- In fact, Toda [1991] shows that with a single call to a #SAT oracle, any problem in the polynomial hierarchy can be solved in polynomial time.
- In this talk, we focus on different proof systems for the propositional model counting problem.
- To be precise, we focus on different proof systems for the language $L = \{(F, k) \mid F \text{ has exactly } k \text{ satisfying assignments}\}$

A Naive Proof Systems for #SAT

- A naive proof system to prove that a CNF formula F has exactly k satisfying assignment is to list the k satisfying assignments along with a resolution proof of the CNF formula F', where F' consist of the following clauses:
  - All clauses of F belongs to F'.
  - For each satisfying assignment α of F, there is a clause $C_\alpha \in F'$, where C_α is the clause that has the unique falsifying assignment α.

A Naive Proof Systems for #SAT

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  - For each satisfying assignment α of F, there is a clause $C_\alpha \in F'$, where C_α is the clause that has the unique falsifying assignment α.
- Example: $(x \vee \overline{y}) \wedge (\overline{x} \vee y)$, satisfying assignments are $\{x = 0, y = 0\}$ and $\{x = 1, y = 1\}$.
  $F' := (x \vee \overline{y}) \wedge (\overline{x} \vee y) \wedge (x \vee y) \wedge (\overline{x} \vee \overline{y})$
  Res-proof of F':

$$\dfrac{\dfrac{(x \vee \overline{y}) \qquad (x \vee y)}{(x)} \qquad \dfrac{(\overline{x} \vee y) \qquad (\overline{x} \vee \overline{y})}{(\overline{x})}}{\bot}$$

MICE Proof System

- Inspired from many #SAT solvers, Fitch et al. SAT-2022, designed a proof system MICE (**Model Counting Induction by Claim Extension**) for #SAT. A simplified and equivalent proof system MICE' is designed by Beyersdorff et al.SAT-2023. These systems work with claims.
- **Claims**: A claim is a 3-tuple (F, A, c), where F is a CNF formula, A is a partial assignment over the vars(F), and c is a count.
- We say that a claim is correct if c is equal to the number of satisfying assignments of $F|_A$.

Definition (MICE, Fitch et al.SAT-2022, Beyersdorff et al.SAT-2023)

A MICE proof of a CNF formula F is a sequence of claims $I_1, I_2, \dots, I_k$ that are derived from inference rules Axiom, Composition, Join, and Extension, such that the final claim I_k is a correct claim of the form $(F, \emptyset, c)$.

Inference rules of MICE

- **Axioms**:

$$\overline{(\emptyset, \emptyset, 1)}$$

Inference rules of MICE

- **Axioms**:

$$\overline{(\emptyset, \emptyset, 1)}$$

- **Composition**:

$$\frac{(F, A_1, c_1) \quad \cdots \quad (F, A_n, c_n)}{(F, A, \sum_{i \in [n]} c_i)}$$

C-1 $\text{vars}(A_1) = \cdots = \text{vars}(A_n)$ and $A_i \neq A_j$ for $i \neq j$.
C-2 $A \subseteq A_i$, for all $i \in [n]$.
C-3 There exists a resolution proof of the CNF formula $A \cup F \cup \{\overline{A_i} \mid i \in [n]\}$.
This proof is called the absence of models statement.

Inference rules of MICE

- **Join**:

$$\frac{(F_1, A_1, c_1) \quad (F_2, A_2, c_2)}{(F_1 \cup F_2, A_1 \cup A_2, c_1 \cdot c_2)}$$

J-1 A_1 and A_2 are consistent.
J-2 $\text{vars}(F_1) \cap \text{vars}(F_2) \subseteq \text{vars}(A_i)$ for $i \in \{1, 2\}$.

Inference rules of MICE

- **Join**:

$$\frac{(F_1, A_1, c_1) \quad (F_2, A_2, c_2)}{(F_1 \cup F_2, A_1 \cup A_2, c_1 \cdot c_2)}$$

J-1 A_1 and A_2 are consistent.
J-2 $\text{vars}(F_1) \cap \text{vars}(F_2) \subseteq \text{vars}(A_i)$ for $i \in \{1, 2\}$.

- **Extension**:

$$\frac{(F_1, A_1, c_1)}{(F, A, c_1 \cdot 2^{|\text{vars}(F) \setminus (\text{vars}(F_1) \cup \text{vars}(A))|})}$$

E-1 $F_1 \subseteq F$,
E-2 $A|_{\text{vars}(F_1)} = A_1$,
E-3 A satisfies $F \setminus F_1$.

- Since all the rules are sound, MICE proof system is sound.

Complexity measures of MICE

- **Size of** π: Let π be a MICE proof of F. Then $s(\pi)$ denotes the size of π which is the total number of claims plus the number of clauses in the resolution proofs in the absence of models statements.
- $c(\pi)$: Another complexity measure is the number of claims in the MICE proof π. This is denoted by $c(\pi)$.

MICE Proof system is Complete

Satisfying Assumption Rule (SA): If A satisfies F, we are allowed to derive the following: $\overline{(F, A, 2^{|\text{vars}(F) \setminus \text{vars}(A)|})}$

Theorem (Fitch et al.SAT-2022, Beyersdorff et al.SAT-2023)

MICE is complete

Proof.

Let F be a CNF formula, and Mod(F) denotes the set of all satisfying assignments of F.
For every assignment $\alpha \in \text{Mod}(F)$, derive $I_\alpha = (F, \alpha, 1)$ via SA.
For all these models, there must be an absence of model statement.
Derive $(F, \emptyset, |\text{Mod}(F)|)$ using the composition rule. □

Corollary (Beyersdorff et al.SAT-2023)

Every CNF formula F has a MICE proof π with $c(\pi) = |Mod(F)| + 2$.

Lower Bounds for MICE

Theorem (Beyersdorff, Hoffmann, Spachmann SAT-2023)

MICE is p-equivalent to Res for unsatisfiable formulas.

- Pigeonhole formulas PHP_n are hard for Res [Haken 1985].

Corollary (Beyersdorff, Hoffmann, Spachmann SAT-2023)

Any MICE proof π of PHP_n has size $s(\pi) = 2^{\Omega(n)}$.

- These lower bounds are not so interesting. As these lower bounds are implied from Res lower bounds.
- PHP_n has a MICE proof π of just one step, i.e., $c(\pi) = 1$.
- Interesting problem: Show lower bounds on the number of claims in the MICE proof.

MICE Lower Bounds on the number of Inference Steps

- Recall that any CNF formula F has a MICE proof π such that $c(\pi) \leq |\mathrm{Mod}(F)| + 2$
- In order to prove the number of claims lower bounds, we must pick CNFs with exponentially many satisfiable assignments.

Definition (XOR PAIRS$_n$)

The formula XOR-PAIRS$_n$ consists of the following clauses:

$$C_{ij}^1 = (x_i \vee x_j \vee \overline{z_{ij}}), \qquad C_{ij}^2 = (\overline{x_i} \vee x_j \vee z_{ij})$$
$$C_{ij}^3 = (x_i \vee \overline{x_j} \vee z_{ij}), \qquad C_{ij}^4 = (\overline{x_i} \vee \overline{x_j} \vee \overline{z_{ij}})$$

for $i, j \in [n]$.

- XOR-PAIRS$_n$ satisfies exactly if $z_{ij} = x_i \oplus x_j$. XOR-PAIRS$_n$ has 2^n models.

MICE Lower Bounds on the number of Inference Steps

Theorem

Any MICE proof π of XOR-PAIRS$_n$ requires claims $c(\pi) = 2^{\Omega(n)}$.

Proof Idea:

- The final claim of a MICE proof must have a large count (i.e., 2^n).
- MICE proof always begin with a small count (i.e., 1).
- In order to reach a large count from a small count with minimum number of steps, a MICE proof must use the Extension or Join steps. Since in these steps, the count gets mutiplied.
- For XOR-PAIRS$_n$, one factor of any such multiplication is always a 1.
- Thus, the only way to increase the count is through the composition rule.
- To reach the count 2^n from 1, exponential number of summands (i.e., composition rules) are required.

Knowledge Compilation

- Next two #SAT proof systems KCPS(#SAT) and CPOG use concepts from knowledge compilation.
- Knowledge compilation has emerged as a new direction of research for dealing with the computational intractable problems like propositional model counting.
- This technique compiles off-line a propositional theory (like CNF formulas) into a target language (like some well studied structures say DNNF).
- The target language (like DNNF) is then used on-line to answer a large number of queries in polynomial time.
- Let us discuss some important target languages used.

Knowledge Compilation: Circuits

- A circuit is a directed acyclic graph with labelled nodes that are called gates. There is a unique gate with in-degree 0, called the root.
- Gate with out-degree 0 are called leaves and are labelled with literals or constant 0 or 1.
- Every inner gate is an AND-, OR-, or NOT-gate and is labelled with the corresponding Boolean function.
- Let D be a circuit. For gates of D, we use uppercase letters such as N.
- $\mathrm{vars}(D)$ denotes the set of all variables that occur in the leaves of D.
- $\mathcal{E}(D)$ denotes a proper encoding of D, where we use a new variable V_N for every gate N.
- $D(N)$ denotes the subcircuit of D with root and N consisting of all descendants of N in D.

Knowledge Compilation: NNF, DNNF, d-DNNF

- A circuit is in **negation normal form** (NNF) if it does not contain NOT-gates.
- An AND-gate with children N_1 and N_2 is called **decomposable**, if $\text{vars}(D(N_1)) \cap \text{vars}(D(N_2)) = \emptyset$.
- An OR-gate with children N_1 and N_2 is called **deterministic** if there is no assignment that satisfies both $D(N_1)$ and $D(N_2)$.
- A **DNNF** (decomposable negation normal form, by Adnan Darwiche, IJCAI-1999), is an NNF where every AND-gate is decomposable.
- A **d-DNNF** (deterministic decomposable NNF, by Adnan Darwiche, JANCL-2001) is a DNNF where every OR-gate is deterministic.

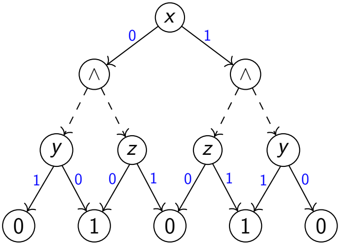
Knowledge Compilation: decision-DNNF

- It is non-trivial to check if all OR-gates are deterministic.
- Decision-DNNF is a restricted version of d-DNNF.
- In a decision-DNNF, any OR-gate has the form $N = (N_1 \text{ or } N_2)$ with $N_1 = (x \text{ and } N_3)$ and $N_2 = (\overline{x} \text{ and } N_4)$ for any variable x.
- Any such OR-gates are deterministic.
- Thus decision-DNNF uses decision gates instead of OR-gates.
- We can assume that the leaves of a Decision-DNNF contain only constants 0 or 1

decision-DNNF

decision-DNNF D over variable set X:

- leaves are either 0 or 1
- decision nodes labeled with x
  - outgoing edges labeled with 0,1
  - x is not repeated in any root-to-leaf path
  - We say that x is tested in D
- decomposable $\wedge$ nodes
- vars(D) = set of variables tested in D

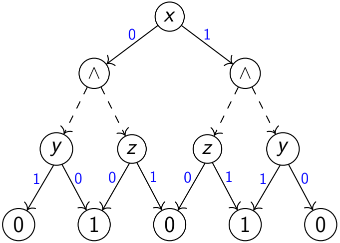


decision-DNNF for $x = y = z$

decision-DNNF

decision-DNNF D **over variable set** X:

- Let $\alpha \in \{0,1\}^X$. A source-sink path P in D is compatible with α if and only if when x is tested on P, the outgoing edge labeled with $\alpha(x)$ is in P.
- We say that α satisfies D, if only 1-gates are reached by paths compatible with α.
- Example: Consider an assignment $\alpha : x = 0, y = 0, z = 1$. α does not satisfy D. Since, it is reaching a 0-gate.

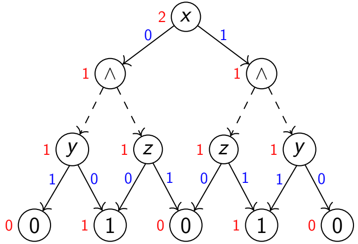


decision-DNNF D for $x = y = z$

Propositional Model Count is easy for decision-DNNF

Theorem (Adnan Darwiche 2001)

Given a decision-DNNF D for a CNF formula F such that $D \equiv F$, it is easy to compute the $|Mod(F)|$.



decision-DNNF D for $x = y = z$

- count models in bottom-up fashion
- assign $0 \rightarrow$ 0-sinks, $1 \rightarrow$ 1-sinks.
- at an $\wedge$-gate: mutiply the model count of both children
- at a decision-gate: let two child nodes be N_1, N_2. Then, model-count $=$ $(2^{|vars(N_1) \setminus vars(N_2)|} \times$ count of $N_2) + (2^{|vars(N_2) \setminus vars(N_1)|} \times$ count of $N_1)$

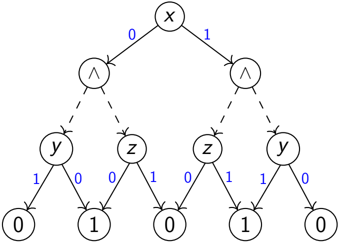
Knowledge Compilation based Proof System for #SAT (KCPS(#SAT))

A KCPS(#SAT) proof of a CNF F provides a decision-DNNF D such that $D \equiv F$.

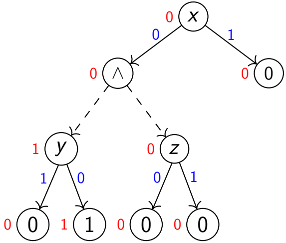
$F := (x \vee \overline{y}) \wedge (\overline{x} \vee y) \wedge (x \vee \overline{z}) \wedge (\overline{x} \vee z)$

Say, $C = (x \vee \overline{z})$. Does decision-DNNF $D \implies C$?

Easy, check if $D \wedge \overline{C}$ is UNSAT.



decision-DNNF D for $x = y = z$



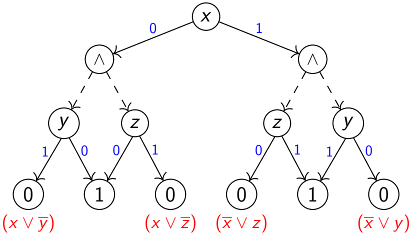
decision-DNNF $D|_{\overline{C}}$

(conditioning D with $\{x = 0, z = 1\}$)

$D \implies F$ ✓

KCPS(#SAT) contd.

$F := (x \vee \overline{y}) \wedge (\overline{x} \vee y) \wedge (x \vee \overline{z}) \wedge (\overline{x} \vee z)$



certified decision-DNNF D for F

Given a decision-DNNF D and a CNF F, it is coNP-complete to check whether $F \implies D$ [Capelli, SAT 2019]

Certified decision-DNNF D over X:

- every 0-sink M has a label of a clause $C_M \in F$ s.t.
  - for every $\alpha \in \{0,1\}^X$, s.t there is a path from source to a 0-sink M compatible with α, we have α falifies C_M.

Let $F(D) :=$ set of all 0-sink labels i.e. $F \implies F(D)$.

$F(D) \implies D$: every α falsifying D ends up at a 0-sink
hence α also falsifies the label $C \in F(D)$ at that sink.

$F \implies D$ ✓

KCPS(#SAT) contd.

Definition (KCPS(#SAT), Capelli, SAT-2019)

Given a CNF F, a certificate that F has exactly k satisfying assignment is a correct certified decision-DNNF D such that:

- every clause of $F(D)$ are clauses of F,
- D computes F and has k satisfying assignments.

For a proof system, the proof must be polynomial time verifiable. The verification process is simple:

- Check that D is correct. That is, check if all labeled clauses at the 0-sinks are correct. This is easy shown by Capelli, SAT-2019.
- Check that $D \equiv F$. Easy
- Check if k satisfying assignment of D. Easy!

Lower bounds for KCPS(#SAT)

- Many lower bounds on the size of decision-DNNFs representing CNFs already known [Paul Beame et al.UAI, 2013, Simone Bova et al., IJCAI,2016]
- For all such CNF formulas, we have KCPS(#SAT) lower bounds.

Theorem (Olaf Beyersdorff et al., SAT-2024)

For unsatisfiable formulas, KCPS(#SAT) *and regular resolution are p-equivalent.*

- All unsatisfiable CNFs which are hard for regular resolution are also hard for KCPS(#SAT)

CPOG: Certified Partitioned Operation Graph

- Another proof system CPOG for #SAT is designed by Randal E. Bryant et al., SAT-2023.
- CPOG is not restricted to the weak certified decision-DNNF, but uses more flexible circuit class POG (Partitioned Operation Graph).
- Model counting is efficient for POG.

Definition (Partitioned Operation Graph)

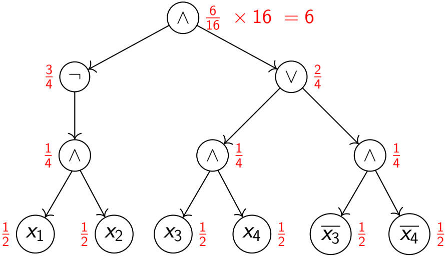
A POG is a d-DNNF with NOT-gates.

- Every AND-gate is decomposable, OR-gate is deterministic, and NOT-gates are allowed in POG.
- Alternatively, a d-DNNF can be viewed as a POG with negation applied only to variables.

Example: Partitioned Operation Graph

$F := (\overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_3} \vee x_4) \wedge (x_3 \vee \overline{x_4})$

An assignment α satisfies the POG if evaluating it on α evaluates to True. For instance, $\{x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1\}$ is a model for P.



POG P for F

Model counting is easy in POG:

- count in bottom-up fashion
- assign $0 \to$ 0-sinks, $1 \to$ 1-sinks, $\frac{1}{2} \to$ literals
- at an $\wedge$-gate: multiply the model count of all children
- at a $\neg$-gate: model count $= 1-$ count at the child node
- at an $\vee$-gate: add the model count of both children
- at root: Final model count $=$ multiply count with $2^{|vars(F)|}$.

CPOG: Certified Partitioned Operation Graph

- Since a proof system must be polynomial time verifiable, any proof system which uses POG to certify #SAT of a CNF formula F must include the following information as well:
  - Encoding $\mathcal{E}(P)$ of the POG P,
  - A proof of the fact that $F \implies P$,
  - A proof of the fact that $P \implies F$,
  - A proof that all the OR-gates used in P are indeed deterministic.
- A POG for a CNF formula F including all the above information is a CPOG proof.

The CPOG representation and the Proof System

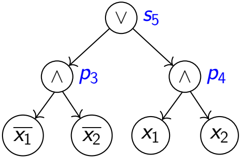
A CPOG proof must contains the following:

- POG representation and clausal encoding of POG $\mathcal{E}(P)$.
- For all OR-gates explicit proof with hints of the fact that they are deterministic.
- For $F \implies P$: The proof contains explicit clause addition steps. A clause can only be added if it is logically implied by the existing clauses. A sequence of clause identifiers must be listed as a hint providing a RUP verification of the implication.
- For $P \implies F$: The proof contains explicit clause deletion steps. A clause can only be deleted if it is logically implied by the remaining clauses. A sequence of clause identifiers must be listed as a hint providing a RUP verification of the implication.

CPOG example (Bryant et al., SAT 2023 slides)

$F := (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2)$

ID	Literals	Explanation
1	1 −2	Input
2	−1 2	Input



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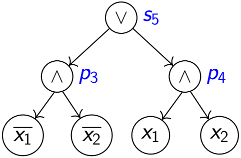


POG Declaration:

Type	Literals	Hint
p	3 −1 −2	-

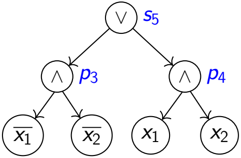
CPOG Proof:

ID	Literals	Explanation
3	3 1 2	p_3
4	−3 −1	
5	−3 −2	

CPOG example (Bryant et al., SAT 2023 slides)

$F := (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2)$

ID	Literals	Explanation
1	1 −2	Input
2	−1 2	Input



POG Declaration:

Type	Literals	Hint
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p	4 1 2	-

CPOG Proof:

ID	Literals	Explanation
3	3 1 2	p_3
4	−3 −1	
5	−3 −2	
6	4 −1 −2	p_4
7	−4 1	
8	−4 2	

CPOG example (Bryant et al., SAT 2023 slides)

$F := (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2)$

ID	Literals	Explanation
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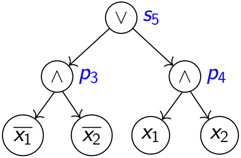
2	−1 2	Input



POG Declaration:

Type	Literals	Hint
p	3 −1 −2	-
p	4 1 2	-
s	5 3 4	4 7 $\rightarrow (\overline{p_3} \vee \overline{p_4})$

CPOG Proof:

ID	Literals	Explanation
3	3 1 2	p_3
4	−3 −1	
5	−3 −2	
6	4 −1 −2	p_4
7	−4 1	
8	−4 2	
9	−5 3 4	s_5, 4 7
10	5 −3	(Input clauses not allowed
11	5 −4	in RUP proof of ∨-gates)

$\rightarrow$ RUP proof of $\overline{p_3} \vee \overline{p_4}$

$\Longrightarrow$ p_3, p_4 have disjoint models

CPOG example (Bryant et al., SAT 2023 slides)

$F := (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2)$

ID	Literals	Explanation
1	1 −2	Input
2	−1 2	Input

RUP Additions:

Type	Literals	Hint
a	−2 5	11 1 6
	↓	↓ ↓ ↓
	2, −5	−4 1 −2

CPOG Proof:

ID	Literals	Explanation
3	3 1 2	p_3
4	−3 −1	
5	−3 −2	
6	4 −1 −2	p_4
7	−4 1	
8	−4 2	
9	−5 3 4	s_5, 4 7
10	5 −3	
11	5 −4	
12	−2 5	11 1 6

CPOG example (Bryant et al., SAT 2023 slides)

$F := (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2)$

ID	Literals	Explanation
1	1 −2	Input
2	−1 2	Input

RUP Additions:

Type	Literals	Hint
a	−2 5	11 1 6
a	5	10 12 2 3
	↓	↓ ↓ ↓ ↓
	−5	−3 −2 −1 3

CPOG Proof:

ID	Literals	Explanation
3	3 1 2	p_3
4	−3 −1	
5	−3 −2	
6	4 −1 −2	p_4
7	−4 1	
8	−4 2	
9	−5 3 4	s_5, 4 7
10	5 −3	
11	5 −4	
12	−2 5	11 1 6
13	5	10 12 2 3

CPOG example (Bryant et al., SAT 2023 slides)

$F := (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2)$

ID	Literals	Explanation
1	1 −2	Input
2	−1 2	Input

RUP deletions:

Type	Clause	Hint
d	12	11 1 6

CPOG Proof:

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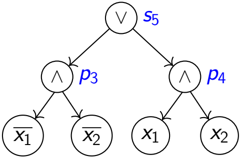

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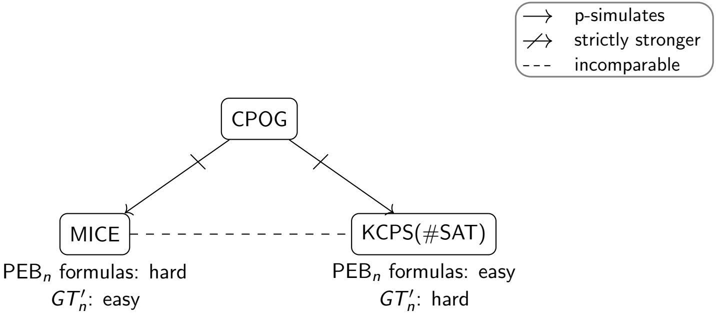
Relationship among #SAT proof systems



- There exists a family of unsatisfiable formulas GT'_n based on the ordering principle which are easy for general Res but are hard for regular Res. [Alekhnovich et al., TOC-2007].
- PEB$_n$ formulas on pyramidal graphs are CNF formulas which are shown to be hard for MICE but easy for KCPS(#SAT) [Beyersdorff et al., SAT-2024].

KCPS and MICE are incomparable

Theorem (Beyersdorff et al.,SAT2024)

KCPS(#SAT) *cannot p-simulate* MICE.

Proof.

- MICE is p-equivalent to Res.
- KCPS(#SAT) is p-equivalent to regular resolution.
- There exists family of CNF formulas which are easy for Res but are hard for regular Res [Alekhnovich et al., TOC-2007].
- Such formulas are easy for MICE but hard for KCPS(#SAT).

□

KCPS and MICE are incomparable

Theorem (Beyersdorff et al., SAT-2024)

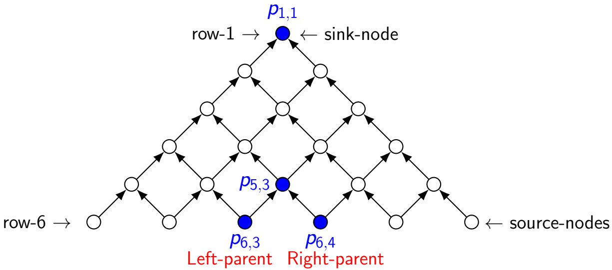
MICE *cannot p-simulate* KCPS(#SAT).

Proof Idea: There exists a family of CNF formulas PEB_n such that it has small certified decision-DNNF D with $D \equiv \text{PEB}_n$ but any MICE proof of PEB_n has size $2^{\Omega(n)}$.

- The CNF formula PEB_n encodes a pebbling game on pyramidal graphs.
- Let us next present the formula PEB_n and an easy KCPS(#SAT) proof for the same.

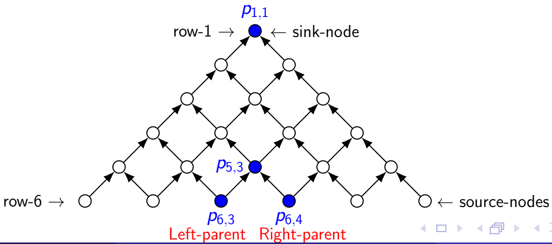
Pyramidal Graph G_n

- PEB_n formulas encode a pebbling game on pyramidal graphs G_n.
- The pyramidal graph G_n has n rows, numbered from 1 to n.
- The row i has i nodes. So, G_n has total $m = \sum_{i=1}^{n} i = n(n+1)/2$ nodes.
- We label each node with $P_{i,j}$, where i corresponds to the row, and j to the column. Clearly, for each node $P_{i,j}$, we have $1 \leq j \leq i \leq n$.
- For each $i < n$, there are edges from $P_{i+1,j}$ and $P_{i+1,j+1}$ to $P_{i,j}$.



PEB_n Formulas

- Before presenting the PEB_n formulas, we briefly discuss the intuition.
- For each node $P_{i,j}$, there are two variables $w_{i,j}$ and $b_{i,j}$.
- $w_{i,j}$ denotes that a white pebble is placed on node $P_{i,j}$.
- $b_{i,j}$ denotes that a black pebble is placed on node $P_{i,j}$.
- PEB_n requires that each source node must contain a pebble (either white or black).
- No node can simultaneously contain a black and a white pebble.
- Every other node needs to contain a pebble if and only if both its parent nodes contain a pebble.



PEB_n Formulas

Definition (PEB_n, Beyersdorff et al.,SAT-2024)

Let n be an integer. The formula PEB_n has variables $w_{i,j}$ and $b_{i,j}$ for every $i,j \in [n]$ with $j \leq i$. The PEB_n is a CNF defined as follows:

– For every $i,j \in [n-1], j \leq i$ the formula requires that

$(w_{i,j} \vee b_{i,j}) \leftrightarrow \Big((w_{i+1,j} \vee b_{i+1,j}) \wedge (w_{i+1,j+1} \vee b_{i+1,j+1})\Big)$ Expressed as:

$$C^1_{i,j} = \overline{w_{i+1,j}} \vee \overline{w_{i+1,j+1}} \vee w_{i,j} \vee b_{i,j} \qquad C^2_{i,j} = \overline{w_{i+1,j}} \vee \overline{b_{i+1,j+1}} \vee w_{i,j} \vee b_{i,j}$$

$$C^3_{i,j} = \overline{b_{i+1,j}} \vee \overline{w_{i+1,j+1}} \vee w_{i,j} \vee b_{i,j} \qquad C^4_{i,j} = \overline{b_{i+1,j}} \vee \overline{b_{i+1,j+1}} \vee w_{i,j} \vee b_{i,j}$$

$$C^5_{i,j} = w_{i+1,j} \vee b_{i+1,j} \vee \overline{w_{i,j}} \qquad C^6_{i,j} = w_{i+1,j} \vee b_{i+1,j} \vee \overline{b_{i,j}}$$

$$C^7_{i,j} = w_{i+1,j+1} \vee b_{i+1,j+1} \vee \overline{w_{i,j}} \qquad C^8_{i,j} = w_{i+1,j+1} \vee b_{i+1,j+1} \vee \overline{b_{i,j}}$$

– For every $i,j \in [n], j \leq i$, there is a clause $C^9_{i,j} = \overline{b_{i,j}} \vee \overline{w_{i,j}}$.

– For every $j \in [n]$, there is a clause $C^{10}_{n,j} = w_{n,j} \vee b_{n,j}$.

PEB$_n$:KCPS(#SAT) proof [Beyersdorff et al.,SAT-2024]

PEB_n:KCPS(#SAT) proof [Beyersdorff et al.,SAT-2024]

$N_{i,j}$: (assuming no pebble at $P_{i,j}$, evaluating bottom-up)

$N_{n,j} \implies 0 \quad (C^{10}_{n,j})$

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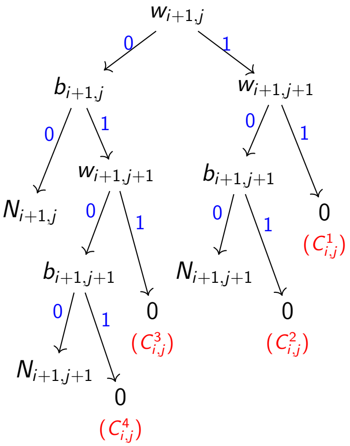

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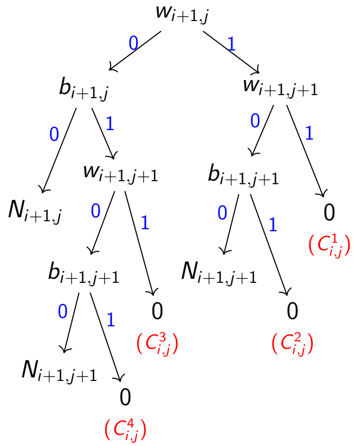
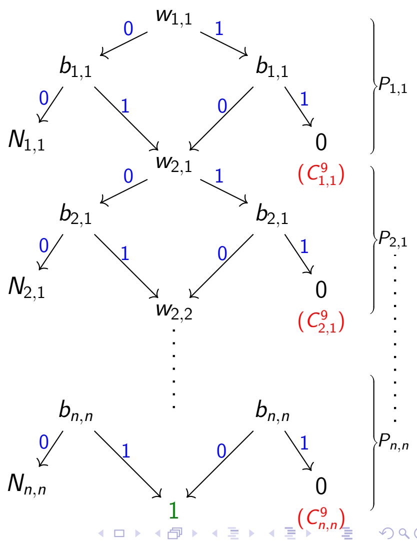
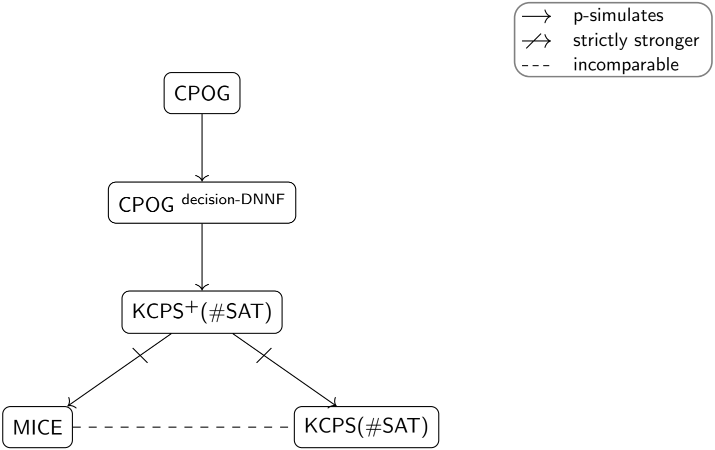


$N_{n,j} \implies 0 \quad (C^{10}_{n,j})$

certified decision-DNNF for PEB_n:



CPOG p-simulates KCPS(#SAT) and MICE



$\mathrm{KCPS}^+(\#\mathrm{SAT})$

- Recall the following definition:

Definition (S-certified Decision DNNF)

Let S be a set of clauses. A decision-DNNF D is called S-certified if every 0-gate N is labelled by a certificate $C \in S$. A clause is a certificate for N if all assignments that reach N falsify C.

Definition ($\mathrm{KCPS}^+(\#\mathrm{SAT})$, Florent Capelli, SAT-2019)

A $\mathrm{KCPS}^+(\#\mathrm{SAT})$ proof of a CNF F is a pair (σ, D) where,

- σ is a Res derivation starting from the clauses in F and
- D is a σ-certified decision DNNF (i.e., all clauses labelling the 0-gates in D are derived in σ) such that $D \equiv F$.

$\text{KCPS}^+(\#\text{SAT})$

Theorem (Beyersdorff et al., SAT-2024)

$\text{KCPS}^+(\#\text{SAT})$ p-simulate KCPS(#SAT).

Proof.

Every KCPS(#SAT) proof D of a CNF formula F can be written as a $\text{KCPS}^+(\#\text{SAT})$ proof (σ, D), where σ contains all clauses from F. □

Theorem (Beyersdorff et al., SAT-2024)

$\text{KCPS}^+(\#\text{SAT})$ p-simulates MICE.

The extraction of the decision-DNNF from a MICE proof was already known. This proof shows how to extract a certified decision-DNNF.

CPOG$^{\text{decision-DNNF}}$

The CPOG$^{\text{decision-DNNF}}$ proof system uses decision-DNNF instead of a POG in the CPOG framework. To be precise,

Definition (CPOG$^{\text{decision-DNNF}}$, Beyersdorff et al., SAT-2024)

A CPOG$^{\text{decision-DNNF}}$ proof of a CNF formula F is a pair $(\mathcal{E}(D), \rho)$ where

- D is a decision-DNNF and $\mathcal{E}(D)$ is a clausal encoding of D such that $D \equiv F$,
- ρ is a proof of $F \implies \mathcal{E}(D)$.

Since decision-DNNF uses decision gates instead of OR-gates, the corresponding proof of CPOG is not needed.
Also, verifying $D \implies F$ is easy.

$\mathrm{CPOG}^{\text{decision-DNNF}}$

Theorem (Beyersdorff et al., SAT-2024)

$\mathrm{CPOG}^{\text{decision-DNNF}}$ p-simulate $\mathrm{KCPS}^{+}(\#\mathrm{SAT})$.

Proof idea: For a CNF formula F, we are given a $\mathrm{KCPS}^{+}(\#\mathrm{SAT})$ proof (σ, D). For the $\mathrm{CPOG}^{\text{decision-DNNF}}$ proof of F, just keep the same decision-DNNF D. Also, using σ it is possible to derive a proof ρ of $F \implies D$.

Theorem (Beyersdorff et al., SAT-2024)

CPOG *p-simulate $\mathrm{CPOG}^{\text{decision-DNNF}}$.*

Proof idea: A decision-DNNF is also a POG.

Conclusion and Open Problems

- In this talk, we discussed about three proof systems for #SAT along with their relationships: MICE, KCPS(#SAT), and CPOG.

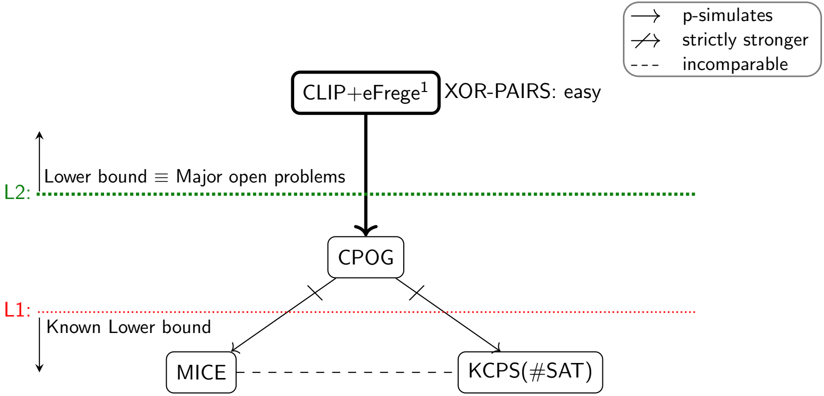
Open Problems:

– Proving lower bounds for POG.

– There exists CNFs with small decision-DNNFs, but requires large certified decision-DNNFs [Beyersdorff et al., SAT-2024]. Similarly, does their exists CNF formulas with small POG, but large CPOG proofs.

– We discussed that XOR-PAIRS formulas are hard for the MICE system. It is open whether XOR-PAIRS formulas are easy or hard for CPOG.

Current #SAT proof complexity



[1]Sravanthi Chede, Leroy Chew, and Anil Shukla. Circuits, Proofs and Propositional Model Counting. FSTTCS 2024.

Thank you.